INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 3398

Avril 1998

THÈME 1

Rapport de recherche

ISSN 0249-6399

INRIA
ROCQUENCOURT

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Thème 1 — Réseaux et systèmes
Projet MEVAL

Rapport de recherche n° 3398 — Avril 1998 — 45 pages

Abstract: $\{\mathcal{P}^{(N)}, N \geq 1\}$ is a sequence of standard polling networks, consisting of N nodes attended by $V^{(N)}$ mobile servers. When a server arrives at a node i, he serves one of the waiting customers, if any, and then moves to node j with probability $p_{ij}^{(N)}$. Customers arrive according to a Poisson process. Service requirements and switch-over times between nodes are independent exponentially distributed random variables. The behavior of $\mathcal{P}^{(N)}$ is analyzed in *thermodynamic limit*, i.e when both N and $V^{(N)}$ tend to infinity, with $U \stackrel{\text{def}}{=} \lim_{N\to\infty} V^{(N)}/N,\ 0 < U < \infty$. First, ergodicity conditions are given. Then, combining the *mean-field* approximation approach together with weak convergence of Markov processes, the joint distribution [customers, vehicles] for an arbitrary finite number of nodes is explicitly characterized. In fact this distribution has a product form, which is the mathematical anologue of *the propagation of chaos*. One also computes the speed of convergence. In most of the study, $\mathcal{P}^{(N)}$ is a fully symmetrical network, but a generalization is carried out for systems provided with only *block-wise* symmetry.

Key-words: Markov process, polling networks, thermodynamic limit, mean-field, propagation of chaos.

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(Résumé : tsvp)

Limite thermodynamique et propagation du chaos dans les réseaux à scrutin

Résumé : $\{\mathcal{P}^{(N)}, N \geq 1\}$ désigne une suite de réseaux à scrutin (*polling*), formés de N noeuds et de $V^{(N)}$ serveurs mobiles. Lorsqu'un serveur arrive à une station i, il sert un éventuel client en attente, puis se dirige vers le noeud j avec probabilité $p_{ij}^{(N)}$. Les arrivées externes de clients forment un processus de Poisson. Les temps de service à chaque noeud, ainsi que les durées de déplacement inter-noeuds sont des variables aléatoires exponentielles, indépendantes.

On analyse le comportement de $\mathcal{P}^{(N)}$ en *limite thermodynamique*, i.e. quand N et $V^{(N)}$ tendent vers l'infini, avec $U \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{V^{(N)}}{N}$, $0 < U < \infty$. D'abord on donne les conditions d'ergodicité. Ensuite, en combinant l'approche approximation *champ moyen*, utilisée en physique statistique, avec la convergence faible pour les processus Markoviens, on caractérise explicitement la distribution jointe [clients, véhicules] pour un nombre fini quelconque de stations. Cette distribution est en fait un produit direct, qui reflète la *propagation du chaos*. On donne également la vitesse de convergence. La plupart de l'étude est réalisée pour des systèmes symétriques, mais une généralisation est donnée dans le cas de symétrie par bloc.

Mots-clé : Processus de Markov, réseaux à scrutin, limite thermodynamique, champ moyen, propagation du chaos.

1 Introduction

1.1 Sketch of the model and preliminaries

Let $\{\mathcal{P}^{(N)}, N \geq 1\}$ be a sequence of standard polling systems, consisting of N nodes attended by $V^{(N)}$ mobile servers. In this paper, we aim at analyzing the behaviour of $\mathcal{P}^{(N)}$ in *thermodynamic limit*, i.e when N and V tend to infinity.

From the abundant literature which has been devoted to polling systems (see e.g. [11]), it emerges that explicit solutions for $\mathcal{P}^{(N)}$ are hard or even impossible to obtain when N is small, even under the simplest assumptions. Also, in many applications (telecommunications, transportation, etc.), it is desirable to understand the behaviour of networks when their size (or volume) increases. In this context, the underlying idea, borrowed from *statistical physics*, will consist in verifying the so-called *chaos hypothesis*. This means, in the symmetrical situation where mean-field interaction applies, that an arbitary finite r-tuple of nodes behaves, when $N \to \infty$, as a set of r independent nodes, each of them being evolving as a time inhomogeneous jump Markov process, whose dynamics can be completely caracterized. One of the salient features is that the transition rates of this process are generally obtained by calculating the empirical measure of all other nodes, which, as $N \to \infty$, becomes deterministic (but time dependent) and is referred to as the *mean-field*. After having solved the problem of weak convergence, the crucial mathematical step is the proof of existence and uniqueness of the mean-field differential equations, which are non-linear.

Incidently, it is worth remarking, that, when the state space is not finite, ergodicity conditions are generally in force (to avoid trivial situations), while, in the finite case, one faces possible *phase transition* phenomenona.

These last years several authors have considered the chaos hypothesis in graphs or networks. We only quote [1, 3, 9], which are recent studies where the reader will find a lot of further references. Roughly, three main approaches exist: diagram estimation techniques (see e.g. [3]) originally employed in statistical physics; the direct approach with generators [1, 12]; coupling and well-posed nonlinear martingale problems [9]. Here, we shall adopt the approach via generators.

In most of our study, $\mathcal{P}^{(N)}$ is a fully symmetric open queueing network, consisting of N nodes (or stations) visited by the $V^{(N)}$ servers. At each node, arrivals of customers form a Poisson process with constant rate λ. Each customer requires service, whose duration is a random variable exponentially distributed with mean $1/\mu$. When a server arrives at a busy node, say i, he serves one of the waiting customers, and then he moves to node j with probability $1/N$. If he reaches a node where there are already as many servers as customers, then he immediately turns towards node j, still with probability $1/N$. The switch-over time τ_{ij} to go from node i to node j is exponentially distributed , with parameter $1/\tau$, $\forall 1 \leq 1 \leq N$. All stochastic input sequences (inter arrival times, services, switch-over times) are supposed to be mutually independent.

In the last section, a more general situation, so-called *block-wise symmetry*, is considered, where $\mathcal{P}^{(N)}$ is divided into K blocks, and typically each block, say i, has N_i nodes and is fully symmetrical.

1.2 Main notation

Due to its density, we choose to gather most of the notational material in this section. Thus as soon as a symbol seems mysterious somewhere, it has a high probability of having being defined here (at least, we hope so !), except in section 2.2, which we wanted to keep more self-contained.

For reasons appearing clearly later on, we shall deal with the typical situation

$$U \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{V^{(N)}}{N}, \quad \text{where } 0 < U < \infty,$$

since the cases $U = 0$ or $U = \infty$ are simpler and of little interest.

Let us introduce now some basic characteristic parameters of the system. Most of the time, the subscripts i, r belong to $\mathcal{S}^{(N)} \stackrel{\text{def}}{=} \{1, \dots, N\}$.

- $q_i^{(N)}(t)$, the number of customers at node i at time t.
- $x_i^{(N)}(t)$, the number of servers at node i at time t.
- $Q^{(N)} \stackrel{\text{def}}{=} \{(q_i^{(N)}(t), x_i^{(N)}(t);\ i \in \mathcal{S}^{(N)}),\ t \geq 0\}$.

- $\alpha_{jk}^{(N,r)}(t) \stackrel{\text{def}}{=} \dfrac{1}{N-r}\#\{i > r,\, q_i^{(N)}(t) = j+k,\ x_i^{(N)}(t) = j\}, \forall j,k, \geq 0$. Physically, $\alpha_{jk}^{(N,r)}(t)$ represents the proportion of queues, among $N-r$ of them, where j customers are in service, and k additional customers are waiting. One always assume that, after completing service, a server chooses its destination node with probability $1/N$.

- $a^{(N)}(\vec{x}) \stackrel{\text{def}}{=} 1 - \dfrac{1}{V^{(N)}}\displaystyle\sum_{i=1}^{N} x_i$ is the proportion of servers in movement in the network $\mathcal{P}^{(N)}$.

- $\rho \stackrel{\text{def}}{=} \dfrac{\lambda}{\mu}$ is the intensity factor.

The forthcoming notation is essentially of mathematical nature.

- $\mathcal{Z}, \mathcal{Z}_+, \mathcal{R}$ are respectively the sets of integers, positive integers and real numbers.

- Vectors (or sequences) will often be written with "overright" arrows (e.g. $\vec{x}$, $\vec{q}^{(N)}$, $\vec{\alpha}^{(N,r)}(t)$). Also, $\vec{e}_j$ stands for the j-th unit vector of $\mathcal{R}^k$, for any k. To render the printing lighter, we shall omit the superscript $^{(N)}$ wherever the meaning is clear from the context.

- δ_z is the Dirac measure concentrated at z, where z is a point or a domain in the Euclidean space, and $\mathbb{1}_{\{A\}}$ is the indicator function of the set A.

- $\mathcal{E} \stackrel{\text{def}}{=} \{(q,x) \in \mathcal{Z}_+^2,\ x \leq q\}$.

- Λ is the family of probability measures $\{\rho_{jk},\ j,k \geq 0\}$ on $\mathcal{Z}_+^2$, subject to the constraint
$$\sum_{j,k\geq 0} j\rho_{jk} \leq U\,.$$

- $\Lambda^{(N,r)}$ denotes the set of the probability measures $\{\rho_{jk},\ j,k \geq 0\}$ on $\mathcal{Z}_+^2$, such that
$$\sum_{j,k\geq 0} j\rho_{jk} \leq \frac{V^{(N)}}{N-r}, \quad \text{for } \rho_{jk} \text{ of the form } \quad \frac{r_{jk}}{N-r},\ r_{jk} \in \mathcal{Z}_+.$$

- E being an arbitrary metric space, $\mathcal{B}(E)$ is the space of bounded real-valued Borel measurable functions on E, $\mathcal{C}(E) \subset \mathcal{B}(E)$ is the Banach subspace of bounded continuous functions, $\mathcal{C}_0(E) \subset \mathcal{C}(E)$ is the subspace of continuous functions vanishing at infinity, $\mathcal{C}_c(E) \subset \mathcal{C}_0(E)$ is the subspace of continuous functions with compact support, and $\mathcal{C}_k(E)$ is the subspace of functions with continuous derivatives up to order $k \geq 1$. In addition, for the sake of shortness, $\mathcal{C}_k$ will stand for $C_k(\mathcal{E}^r \times \Lambda)$, $k \geq 0$. Also, we will write $\mathcal{C}^+(E)$ or $\mathcal{B}^+(E)$ when only positive functions are considered. As usual, the metric is induced by the norm $\|f\|=\sup_{x\in E} f(x)$.
- For any complete separable metric space S, let $D_S[0,\infty]$ denote the space of right continuous functions $f : [0,\infty] \to S$ with left limits, endowed with the Skorokhod topology.

2 Prologue and outcomes

Basically two categories of systems will be analyzed.

1 Symmetrical networks

2 Block-wise symmetrical networks

The results are completely proved in case 1 – the extensions being straightforward and only sketched for systems of type 2 –, and they are valid under *arbitrary* initial conditions. Nonetheless, we chose, for the ease of exposition, to split the initial conditions into two categories.

- Semi-general initial conditions, where at time $t = 0$, customers are arbitrarily spread over stations, but all vehicules select their destination with probability $1/N$;
- General initial conditions. For instance, a number of vehicules are *en route* to some specified station.

2.1 Symmetrical system

To begin with, we quote the following theorem, which ensures the problem to be, in some sense, meaningful.

Theorem 2.1 *$\mathcal{P}^{(N)}$ is ergodic if, and only if,*

$$\rho + \lambda\tau < \frac{V^{(N)}}{N}. \tag{2.1}$$

2.1.1 Semi-general initial conditions

From its very definition, one can see that $Q^{(N)}$ is a Markov process and its generator $G^{(N)}$ is the operator given by

$$\begin{aligned} G^{(N)} f(\vec{q},\vec{x}) = \lambda \sum_{i=1}^{N} \big(f(\vec{q}+\vec{e}_i,\vec{x}) - f(\vec{q},\vec{x})\big) \\ + \mu \sum_{i=1}^{N} x_i \big(f(\vec{q}-\vec{e}_i,\vec{x}-\vec{e}_i) - f(\vec{q},\vec{x})\big) \\ + \frac{V^{(N)} a^{(N)}(\vec{x})}{\tau N} \sum_{i=1}^{N} \mathbb{1}_{\{x_i<q_i\}} \big(f(\vec{q},\vec{x}+\vec{e}_i) - f(\vec{q},\vec{x})\big), \end{aligned}$$

where $f \in \mathcal{B}(\mathcal{E}^N)$ and $(\vec{q},\vec{x}) \in \mathcal{E}^N$.
In fact $G^{(N)}$ itself is not easy to manipulate and we proceed in a different way. Assuming inequality (2.1) to be fulfilled for all $N \leq \infty$, which implies also

$$\rho + \lambda\tau < U, \tag{2.2}$$

it will be shown that the joint stationary distribution of any arbitrary finite set of r nodes is equivalent, when $N \to \infty$, to the product of r identical laws (which can be explicitly computed), i.e. the r nodes are asymptotically independent. In other words, one can characterize the limit behaviour of the process

$$\vec{Y}^{(N,r)} \stackrel{\text{def}}{=} \Big\{ \big((q_1^{(N)}(t), x_1^{(N)}(t)), \ldots, (q_r^{(N)}(t), x_r^{(N)}(t)), \vec{\alpha}^{(N,r)}(t)\big),\ t \geq 0 \Big\}, \tag{2.3}$$

where the value of $\vec{Y}^{(N,r)}$ at time t will be written $\vec{Y}^{(N,r)}(t)$. In fact, due to the symmetry of the network, $Y^{(N,r)}$ is a Markov process, whose generator $G^{(N,r)}$ satisfies

$$
\begin{aligned}
G^{(N,r)}f(\vec{q},\vec{x},\vec{\alpha}) &= \lambda\sum_{i=1}^{r}(f(\vec{q}+\vec{e}_i,\vec{x},\vec{\alpha})-f(\vec{q},\vec{x},\vec{\alpha})) \qquad (2.4)\\
&+ \mu\sum_{i=1}^{r}x_i(f(\vec{q}-\vec{e}_i,\vec{x}-\vec{e}_i,\vec{\alpha})-f(\vec{q},\vec{x},\vec{\alpha}))\\
&+ \frac{b^{(N)}(\vec{x},\vec{\alpha})}{N\tau}\sum_{i=1}^{r}\mathbb{1}_{\{x_i<q_i\}}(f(\vec{q},\vec{x}+\vec{e}_i,\vec{\alpha})-f(\vec{q},\vec{x},\vec{\alpha}))\\
&+ \lambda(N-r)\sum_{j,k\geq 0}\alpha_{jk}\Big(f\Big(\vec{q},\vec{x},\vec{\alpha}+\tfrac{\vec{e}_{j,k+1}}{N-r}-\tfrac{\vec{e}_{jk}}{N-r}\Big)-f(\vec{q},\vec{x},\vec{\alpha})\Big)\\
&+ \mu(N-r)\sum_{j,k\geq 0}j\alpha_{jk}\Big(f\Big(\vec{q},\vec{x},\vec{\alpha}+\tfrac{\vec{e}_{j-1,k}}{N-r}-\tfrac{\vec{e}_{jk}}{N-r}\Big)-f(\vec{q},\vec{x},\vec{\alpha})\Big)\\
&+ \frac{(N-r)b^{(N)}(\vec{x},\vec{\alpha})}{\tau N}\sum_{j\geq 0,k>0}\alpha_{jk}\Big(f\Big(\vec{q},\vec{x},\vec{\alpha}+\tfrac{\vec{e}_{j+1,k-1}}{N-r}-\tfrac{\vec{e}_{jk}}{N-r}\Big)-f(\vec{q},\vec{x},\vec{\alpha})\Big),
\end{aligned}
$$

where $f\in\mathcal{B}(\mathcal{E}^r\times\Lambda^{(N,r)})$, $(\vec{q},\vec{x},\vec{\alpha})\in\mathcal{E}^r\times\Lambda^{(N,r)}$, and

$$b^{(N)}(\vec{x},\vec{\alpha}) = V^{(N)}-\sum_{i=1}^{r}x_i-(N-r)\sum_{j,k\geq 0}j\alpha_{jk}\,.$$

It appears that $G^{(N,r)}\stackrel{\text{def}}{=}A^{(N,r)}+B^{(N,r)}$ is the sum of two operators (not explicitly written for the sake of brevity):

- $A^{(N,r)}$, which is equal to the sum of the first three terms in the right-hand side of (2.4), concerns the r isolated queues;
- $B^{(N,r)}=G^{(N,r)}-A^{(N,r)}$, which refers to the evolution of the mean-field empirical measure $\vec{\alpha}^{(N,r)}(t)$.

One will show that

$$\lim_{N\to\infty}G^{(N,r)}=\lim_{N\to\infty}(A^{(N,r)}+B^{(N,r)})\stackrel{\text{def}}{=}G\stackrel{\text{def}}{=}A+B,$$

where A, B (defined below) and G are operators with respective domains $\mathcal{D}(A), \mathcal{D}(B)$ and $\mathcal{D}(G) = \mathcal{D}(A) \cap \mathcal{D}(B)$, with the important property that the closure

$$\overline{G} = A + \overline{B}$$

is the generator of a positive contraction semigroup.

Letting, for any $\vec{\theta} \in \Lambda$, $\theta = 1 - \dfrac{1}{U}\sum_{j,k\geq 0} j\theta_{jk}$, we have

$$\begin{aligned}
Ah(\vec{q},\vec{x},\vec{\theta}) &\stackrel{\text{def}}{=} \lambda \sum_{i=1}^{r} \big(h(\vec{q}+\vec{e}_i,\vec{x},\vec{\theta}) - h(\vec{q},\vec{x},\vec{\theta})\big) \\
&\quad + \mu \sum_{i=1}^{r} x_i \big(h(\vec{q}-\vec{e}_i,\vec{x}-\vec{e}_i,\vec{\theta}) - h(\vec{q},\vec{x},\vec{\theta})\big) \\
&\quad + \frac{\theta U}{\tau} \sum_{i=1}^{r} \mathbb{1}_{\{x_i<q_i\}} \big(h(\vec{q},\vec{x}+\vec{e}_i,\vec{\theta}) - h(\vec{q},\vec{x},\vec{\theta})\big), \quad \forall h \in \mathcal{C}_0, \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
Bf(\vec{q},\vec{x},\vec{\theta}) &\stackrel{\text{def}}{=} \lambda \sum_{j,k\geq 0} \theta_{jk} \left(\frac{\partial f}{\partial \theta_{j,k+1}} - \frac{\partial f}{\partial \theta_{jk}}\right) \\
&\quad + \mu \sum_{j,k\geq 0} j\theta_{jk} \left(\frac{\partial f}{\partial \theta_{j-1,k}} - \frac{\partial f}{\partial \theta_{jk}}\right) \\
&\quad + \frac{\theta U}{\tau} \sum_{j\geq 0, k>0} \theta_{jk} \left(\frac{\partial f}{\partial \theta_{j+1,k-1}} - \frac{\partial f}{\partial \theta_{jk}}\right), \ \forall f \in \mathcal{C}_1. \qquad (2.6)
\end{aligned}$$

One of the basic results of our study claims that, when $N \to \infty$, the empirical measure $\vec{\alpha}^{(N,r)}(t)$ tends to become a deterministic dynamical system *denoted by* $\vec{\alpha}(t)$, which evolves according to the following infinite system of nonlinear differential equations:

$$\begin{aligned}
&\frac{d\alpha_{j,k}(t)}{dt} + \big[\lambda + \alpha(t)\mathbb{1}_{\{k>0\}} + j\mu\big]\alpha_{j,k}(t) = \\
&\qquad \lambda\alpha_{j,k-1}(t) + \alpha(t)\alpha_{j-1,k+1}(t) + \mu(j+1)\alpha_{j+1,k}(t), \quad \forall j,k \geq 0, \qquad (2.7)
\end{aligned}$$

where $\vec{\alpha}(t) \in \Lambda, \ \forall t \geq 0$, and

$$\alpha(t) \stackrel{\text{def}}{=} \frac{U}{\tau}\Big(1 - \frac{1}{U}\sum_{j,k\geq 0} j\alpha_{jk}(t)\Big)\,. \tag{2.8}$$

More precisely, let $\vec{\alpha}(t)$ be a solution of (2.7), (2.8), with initial condition $\vec{\alpha}(0) \in \Lambda$, and define the one-parameter family $\{S(t),\, t \geq 0\}$ of operators on $\mathcal{C}_0$ by

$$S(t)f(\vec{q},\vec{x},\vec{\alpha}(0)) = f(\vec{q},\vec{x},\vec{\alpha}(t)). \tag{2.9}$$

It is immediate to check that $\{S(t),\ t \geq 0\}$ is a strongly continuous positive contraction semigroup, which is generated by $\overline{B}$, the closure of B.

The global situation is pictured in the next three theorems.

Theorem 2.2

(i) *For each $t \geq 0$, there exists a unique distribution $\vec{\alpha}(t)$ satisfying the non-linear system (2.7), (2.8).*

(ii) *In addition, if the ergodicity condition $\rho + \lambda\tau < U$ [see (2.2)] is satisfied, then $\lim_{t\to\infty} \vec{\alpha}(t) = \vec{\pi} > \vec{0}$, where*

$$\pi_{jk} = \frac{e^{-\rho}(1-\beta)\beta^k \rho^j}{j!}, \quad \forall j,k \geq 0, \tag{2.10}$$

with

$$\beta \stackrel{def}{=} \frac{\lambda}{\alpha} \quad \text{and} \quad \alpha \stackrel{def}{=} \lim_{t\to\infty} \alpha(t) = \frac{U-\rho}{\tau}\,. \tag{2.11}$$

Theorem 2.3 *Let $\vec{\alpha}(t)$ be a solution of (2.7), (2.8) and assume $Y^{(N,r)}(0)$ converges weakly to the product distribution $\nu^{(1)} \otimes \ldots \times \nu^{(r)} \otimes \delta_{\vec{\alpha}(0)}$ in $\mathcal{E}^r \times \Lambda$. Then the process $\vec{Y}^{(N,r)}(t)$ converges weakly in $D_{\mathcal{E}^r\times\Lambda}[0,\infty]$ to the product distribution $P_t^{(1)} \otimes \ldots \otimes P_t^{(r)} \otimes \delta_{\vec{\alpha}(t)}$, where, for any $1 \leq l \leq r$, $P_t^{(l)}$ denotes the probability distribution of the $\mathcal{E}$-valued time inhomogeneous Markov process*

$\big(\tilde{q}^{(l)}(t), \tilde{x}^{(l)}(t), t \geq 0\big)$, *with initial distribution* $\nu^{(l)}$, *and whose transition rates from state* (i,j) *to state* (k,m) *at time* t *are given by*

$$\gamma(t;(i,j),(k,m)) \stackrel{def}{=} \begin{cases} \lambda, & \text{for } k = i+1,\ m = j, \\ j\mu, & \text{for } k = i-1,\ m = j-1, \\ \alpha(t)\mathbb{1}_{\{i>j\}}, & \text{for } k = i,\ m = j+1. \end{cases} \tag{2.12}$$

The generator of the process $\lim_{N\to\infty} \vec{Y}^{(N,r)}$ *is* $G = A + B$, *given by (2.5), (2.6), and the forward Kolmogorov's equations associated to* $(\tilde{q}^{(l)}(t), \tilde{x}^{(l)}(t))$ *differ only notationally from (2.7), so that* $P_t^{(l)}$ *and* $\vec{\alpha}(t)$ *are* distinguishable, *so to speak, only by their initial conditions.*

The next proposition concerns the steady state analysis.

Theorem 2.4

$$\lim_{t\to\infty} \lim_{N\to\infty} Y^{(N,r)}(t) \Rightarrow \lim_{N\to\infty} \lim_{t\to\infty} Y^{(N,r)}(t) \Rightarrow \vec{\pi}^{\otimes r} \otimes \delta_{\vec{\pi}}\,,$$

where the symbol " $\Rightarrow$ " denotes here equality in distribution and $\vec{\pi}$ *is the measure given by (2.10).*

2.1.2 General initial conditions

When at time $t = 0$, a part (or even all of them) of the servers are in transit between given nodes, it is obviously not correct to say they choose their destination with a uniform probability $1/N$. Now the process $\vec{Y}^{(N,r)}$ does not give a full account of the evolution of the system, which requires to introduce, as part of the state variables, the vector of vehicles in transit, and hence the mean-field vector would be a 3-dimensional random walk. Although feasible, this task would not be very pleasant, neither for the authors, nor for the reader. Luckily enough, the extension of the results about steady state, obtained in section 2.1.1, can be achieved. Indeed, the argument is based on the existence of a coupling with systems working under semi-symmetrical conditions.

Theorem 2.5 *Let* $\vec{Z}^{(N,r)}(0)$ *denote a finite, but otherwise arbitrary, initial state of the system* $\mathcal{P}^{(N)}$. *If* $\vec{Z}^{(N,r)}(0)$ *converges weakly, when* $N \to \infty$, *to a product distribution, then all conclusions of theorems 2.2 and 2.4 are still valid.*

2.2 The block-wise symmetrical case

In this section $\mathcal{P}^{(N)}$ has only a blockwise symmetry. More precisely, for all N, $\mathcal{P}^{(N)}$ is divided into K blocks, numbered from 1 to K, and block i is assumed to contain $N_i^{(N)}$ nodes.

Thus, setting $\mathcal{K} \stackrel{\text{def}}{=} \{1, \dots, K\}$, we have $\displaystyle\sum_{i\in\mathcal{K}} N_i^{(N)} = N$. In addition, each block operates as a symmetrical polling system of the type analyzed in the preceding section:

- Arrivals form a Poisson process with rate λ_i and service times of customers are exponential random variables, with mean $1/\mu_i$.
- When a server (vehicle) arrives at a node in block i, he serves one of the waiting customers, if any. Then he selects a block j with probability p_{ij}, and, inside block j, the destination node is finally chosen with probability $1/N_j^{(N)}$. The p_{rs}'s, $\forall\, r, s \in \mathcal{K}$, are elements of an ergodic matrix, the invariant measure of which will be denoted by $\pi = (\pi_1, \dots, \pi_K)$.
- The time to go from a node in block i to a node in block j is exponentially distributed, with parameter $1/\tau_{ij}$, $\forall 1 \le i, j \le K$, independently of N.

Assuming all stochastic input sequences to be mutually independent, we introduce – at the expense of a slightly heavier notation valid only inside this section – further characteristic parameters of the systems.

- $q_l^{(N,r)}(t)$, the number of customers at node l in block r at time t.
- $x_l^{(N,r)}(t)$, the number of servers at node l in block r at time t.
- $\alpha_{jk}^{(N,r)}(t) \stackrel{\text{def}}{=} \dfrac{1}{N_r^{(N)}} \#\{l,\ q_l^{(N,r)}(t) = j+k,\ x_l^{(N,r)}(t) = j\}$, for all $j, k \ge 0$ and any arbitrary block r.
- $\overline{\tau} \stackrel{\text{def}}{=} \displaystyle\sum_{r,s\in\mathcal{K}} \pi_r p_{rs} \tau_{rs}$.
- $\rho_r \stackrel{\text{def}}{=} \dfrac{\lambda_r}{\mu_r}$, the intensity factor in block r, and $\overline{\rho}^{(N)} \stackrel{\text{def}}{=} \displaystyle\sum_{r\in\mathcal{K}} \rho_r N_r^{(N)}$.

- $U_i \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{V^{(N)}}{N_i^{(N)}}$, with $0 < U_i < \infty$, and $U \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{V^{(N)}}{N}$.

Like in the fully symmetrical case, it can be shown that the empirical measure $\vec{\alpha}^{(N)}(t)$, as $N \to \infty$, becomes a deterministic dynamical system, again denoted by $\vec{\alpha}(t)$, which evolves according to the following system of nonlinear differential equations.

$$
\begin{aligned}
&\frac{d\alpha_{j,k}^{(r)}(t)}{dt} + \left[\lambda_r + \eta_r(t)\mathbb{1}_{\{k>0\}} + j\mu_r\right]\alpha_{j,k}^{(r)}(t) = \\
&\quad \lambda_r \alpha_{j,k-1}^{(r)}(t) + \eta_r(t)\alpha_{j-1,k+1}^{(r)}(t) + \mu_r(j+1)\alpha_{j+1,k}^{(r)}(t), \quad \forall j,k \geq 0, \quad \forall r \in \mathcal{K},
\end{aligned}
\tag{2.13}
$$

where the $\eta_{rs}(t)$'s satisfy the system

$$
\tau_{rs}\frac{d\eta_{rs}(t)}{dt} + \eta_{rs}(t) = \frac{p_{rs}U_s}{U_r}\left[\mu_r M_r(t) + \eta_r(t)J_r(t)\right], \quad \forall r,s \in \mathcal{K}, \tag{2.14}
$$

with

$$
\eta_s(t) = \sum_{r\in\mathcal{K}} \eta_{rs}(t), \quad M_r(t) = \sum_{j,k\geq 0} j\alpha_{j,k}^{(r)}(t), \quad J_r(t) = \sum_{j\geq 0} \alpha_{j,0}^{(r)}(t),
$$

and the conservation law

$$
\sum_{r,s\in\mathcal{K}} \frac{\tau_{rs}\eta_{rs}(t)}{U_s} + \sum_{r\in\mathcal{K}} \frac{M_r(t)}{U_r} = 1.
$$

Then the propagation of chaos still takes place and follows mainly from the propositions stated below, which deal with the behaviour of the empirical measure $\vec{\alpha}^{(N)}(t)$.

Theorem 2.6 $\mathcal{P}^{(N)}$ *is ergodic if, and only if,*

$$
\frac{\lambda_s \overline{\tau}}{V^{(N)} - \overline{\rho}^{(N)}} < \frac{\pi_s}{N_s}, \quad \forall s \in \mathcal{K}. \tag{2.15}
$$

Theorem 2.7

(i) *For each $t \geq 0$, there exists a unique distribution $\vec{\alpha}(t)$ satisfying the differential systems (2.13), (2.14).*

(ii) *Under the ergodicity condition (2.15), we have*

$$\lim_{t \to \infty} \vec{\alpha}(t) = \vec{\psi} > \vec{0},$$

where

$$\psi_{jk}^{(r)} = \frac{e^{-\rho_r}(1-\beta_r)\beta_r^k \rho_r^j}{j!}, \quad \forall j, k \geq 0, \quad \forall r \in \mathcal{K},$$

with

$$\beta_r \stackrel{def}{=} \frac{\lambda_r}{\eta_r} \quad \text{and} \quad \eta_r \stackrel{def}{=} \lim_{t \to \infty} \eta_r(t) = \left(1 - \sum_{i=1}^{K} \frac{\rho_i}{U_i}\right) \frac{U_r \pi_r}{\overline{\tau}}.$$

Theorem 2.8 *Let $\vec{\alpha}(t)$ be a solution of (2.13), (2.14), and suppose $\vec{\alpha}^{(N)}(0)$ converges weakly to $\delta_{\vec{\alpha}(0)}$. Then $\vec{\alpha}^{(N)}(t)$ converges weakly in $D[0,\infty]$ to $\delta_{\vec{\alpha}(t)}$ and*

$$\lim_{t \to \infty} \lim_{N \to \infty} \vec{\alpha}^{(N)}(t) \Rightarrow \lim_{N \to \infty} \lim_{t \to \infty} \vec{\alpha}^{(N)}(t) \Rightarrow \delta_{\vec{\psi}},$$

where $\vec{\psi}$ is given in theorem 2.6.

3 Technicalities

All proofs of the theorems listed in section 2.1 are derived hereafter. As it will emerge, the extension to the block-wise symmetrical case of section 2.2 is only of technical order, without theoretical difficulties, and consequently will be omitted.

3.1 Ergodicity conditions and tightness

This paragraph contains the proof of theorem 2.1. It also shows the tightness of the family of invariant measures associated to the processes $\vec{Y}^{(N,r)}$, defined in (2.3). First, let us introduce the following quantities:

- $\Omega_i^{(N)}(t)$, the total amount of work arrived at node i up to time t;

- $W_i^{(N)}(t)$, the load at node i, at time t;

Then, pathwise,

$$W_i^{(N)}(t) = \Omega_i^{(N)}(t) - \int_0^t x_i^{(N)}(s)ds. \tag{3.1}$$

To show the necessity of condition (2.1), suppose $\mathcal{P}^{(N)}$ is ergodic. Then, applying the standard ergodic theorem for Markov processes, in (3.1), we can write

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t G^{(N)} f(\vec{q}^{(N)}(s), \vec{x}^{(N)}(s))ds = 0 \quad \text{a.s.}\,, \tag{3.2}$$

for f belonging to the domain of the generator $G^{(N)}$, which was introduced in section 1.2, and also

$$\lim_{t\to\infty} \frac{\Omega_i^{(N)}(t)}{t} = \rho \quad \text{a.s.} \tag{3.3}$$

Consequently,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i^{(N)}(s)ds = \rho \quad \text{a.s.} \tag{3.4}$$

Choosing $f(\vec{q},\vec{x}) = V^{(N)}a^{(N)}(\vec{x}) = V^{(N)} - \sum_{i=1}^N x_i$, for all $(\vec{q},\vec{x}) \in \mathcal{E}^N$, we have

$$G^{(N)} f(\vec{q},\vec{x}) = \mu\sum_{i=1}^N x_i - \frac{V^{(N)}a^{(N)}(\vec{x})}{\tau}\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{x_i<q_i\}}\,. \tag{3.5}$$

Instantiating (3.5) in (3.2) and using

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{V^{(N)}a^{(N)}(\vec{x})}{\tau N}\sum_{i=1}^N \mathbb{1}_{\{x_i^{(N)}(s)=q_i^{(N)}(s)\}}ds > 0 \quad \text{a.s.}\,,$$

the relations (3.2), (3.4) (3.5) imply inequality (2.1).

In order to prove the sufficiency, consider an arbitrary node i, for some fixed initial condition at time 0. It can be viewed as a special queue, with $V^{(N)}$

autonomous servers working in parallel. Then the equivalent service time of each server is dominated in distribution by a random variable D which has the form

$$D = \Delta_1 + \Delta_2 \cdots + \Delta_{Z+1},$$

where

- Z is geometrically distributed with parameter $\frac{N-1}{N}$;
- the random variables Δ_i are independent, with the same distribution as $\Delta \stackrel{\text{def}}{=} R + S$, where R and S are independent exponential variables, with respective parameters μ and $1/\tau$.

Assume node i contains at $t = 0$ exactly one customer in excess (i.e. waiting for a server), and let T represent the first time (a kind of *busy period*) until there is no customer in excess at that node. Then the following inequality holds

$$P(T > t) \leq P\Big(\mathcal{P}_\lambda(t) \geq \sum_{i=1}^{V} N_i(t)\Big), \tag{3.6}$$

where $\mathcal{P}_\lambda(t)$ is a Poisson process of parameter λ and the $N_i(t)$'s, $i \geq 1$, are independent identically distributed renewal point processes, with interrenewal epochs distributed as D. Just for convenience in the proof, the following symbols will be used:

- $D(t)$, the distribution function of D and $d(t)$ its density;
- $H_\gamma(t) = E\big(e^{-\gamma N_i(t)}\big)$;
- $p = \frac{1}{N}$, $q = 1 - p$, $a = \mu + \frac{1}{\tau}$, $b = \frac{\mu}{\tau}$;
- $f^*(s) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-st} f(t)dt$, the ordinary Laplace transform of f.

It follows from Tchebychev's inequality that, for any number $\gamma > 0$,

$$P(T > t) \leq e^{\lambda t(e^\gamma - 1)} H_\gamma^V(t). \tag{3.7}$$

Now $H_\gamma(t)$ does satisfy the renewal equation

$$H_\gamma(t) = 1 - D(t) + e^{-\gamma}\int_0^t H_\gamma(t-s)d(s)ds.$$

Taking Laplace transform and using

$$d^*(s) = \frac{pb}{s^2 + as + pb},$$

we get, after a straightforward computation,

$$H^*_\gamma(s) = \frac{a+s}{s^2 + as + pb(1-e^{-\gamma})} \stackrel{\text{def}}{=} \frac{1}{s_1 - s_0}\left(\frac{a - s_0}{s + s_0} - \frac{a - s_1}{s + s_1}\right),$$

where s_0, s_1, $0 < s_0 < s_1$, are the moduli of the roots of the second degree equation

$$s^2 + as + pb(1 - e^{-\gamma}) = 0,$$

given by

$$s_0 = \frac{a - \sqrt{a^2 - 4pb(1-e^{-\gamma})}}{2}, \quad s_1 = \frac{a + \sqrt{a^2 - 4pb(1-e^{-\gamma})}}{2}.$$

Inverting $H^*_\gamma(s)$, we obtain

$$H_\gamma(t) = e^{-s_0 t}\left[1 + \frac{s_0}{s_1 - s_0}\left(1 - e^{-(s_1 - s_0)t}\right)\right].$$

Since

$$s_0 = \frac{pb(1-e^{-\gamma})}{a} + O(p^2),$$

(here $O(p^2)$ is in fact a positive function) and, ex hypothesis, the product pV is bounded for all N, it follows that

$$H^V_\gamma(t) \le Me^{-tVs_0}, \quad \forall t, N \ge 0,$$

whence

$$P(T > t) \le \exp\left[\lambda(e^\gamma - 1)t - \frac{tVp(1-e^{-\gamma})(1 + o(Vp))}{\frac{1}{\mu} + \tau}\right]. \tag{3.8}$$

Now, under condition (2.1), it is always possible to pick $\gamma > 0$ in (3.8), to ensure

$$P(T > t) \leq Ke^{-\gamma t}, \quad \forall t \geq 0,\ \forall N,$$

where K is a constant independent of N.

Using elementary properties of the Poisson process together with Little's formula, remembering that i is arbitrary and assuming (2.1) holds, one can state two conclusions – provided that the initial number of customers at each node is bounded, this being in no way a restriction – :

- $\mathcal{P}^{(N)}$ is ergodic;
- the mean number of customers waiting at an arbitrary node remains uniformly bounded with respect to N; this property is crucial for theorem 2.4. ■

3.2 The dynamical system $\vec{\alpha}(t)$ and theorem 2.2

In this section, we shall prove existence an uniqueness of a solution of system (2.7), under the sole assumption that $\vec{\alpha}(0)$ is finite. The key point is to remark that (2.7) can be viewed as *forward* Kolmogorov's equations of a Markov process denoted by $\xi(t)$, which depict the evolution of the tandem queue network shown in figure 3.1. The transition rates of the underlying two-dimensional random walk drawn in figure 3.2 are not homogeneous in space or time.

– The first queue is of $\mathrm{M/M_t/1/\infty}$ type, with $FIFO$ discipline.

– The second one is an $\mathrm{M_t/M/\infty}$ infinite server queue.

Keeping in mind the above interpretation, the line of reasoning will be the following one.

STEP A First, show existence and uniqueness of a bounded continuous and strictly positive function $\alpha(t)$, $\forall t \geq 0$, satisfying (2.8), the Gordian knot being the existence of $\lim_{t\to\infty} \alpha(t) = \alpha$.

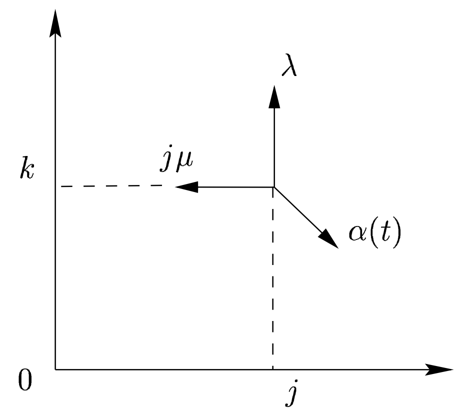

Figure 3.1: Embedded time-inhomogeneous random walk

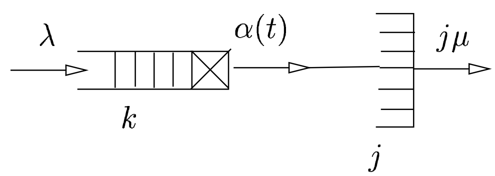

Figure 3.2: Equivalent queueing network

STEP B Then, deduce (from the general theory) that system (2.7) has a unique probabilistic solution for all $t < \infty$, and that, under condition (2.2), there exists a unique stationary distribution for the tandem queue network.

Remark Due to the constraint (2.8), the system (2.7) is not linear, but quadratic. Readers belonging to the control theory community can discern here a control problem in open loop, which in principle has connections with Ricatti's differential equations...

3.2.1 STEP A

Introduce the generating function

$$F(x,y,t) \stackrel{\text{def}}{=} \sum_{j,k\geq 0} \alpha_{jk}(t) x^j y^k,$$

where x, y are complex variables such that $0 \leq |x|, |y| \leq 1$. Hence, (2.7) and (2.8) are equivalent to the non-linear functional equation

$$\frac{\partial F(x,y,t)}{\partial t} + \left(\lambda(1-y) + \alpha(t)\Big(1 - \frac{x}{y}\Big)\right) F(x,y,t) = \\ \mu(1-x)\frac{\partial F(x,y,t)}{\partial x} + \alpha(t)\Big(1-\frac{x}{y}\Big)F(x,0,t), \quad (3.9)$$

where, according to (2.8),

$$\alpha(t) = \frac{U}{\tau}\Big(1 - \frac{1}{U}\frac{\partial F(1,1,t)}{\partial x}\Big). \quad (3.10)$$

$F(x,y,t)$ is sought to be holomorphic in the region $0 \leq |x|, |y| < 1$, continuous in $0 \leq |x|, |y| \leq 1$ with, in particular,

$$F(1,1,t) = 1, \quad \forall t < \infty\,.$$

The physical constraints of the problem, expressed by equations (2.8) or (3.10), show $\alpha(t)$ is positive and uniformly bounded by U/τ for all t. Consequently, the number of units in the $\mathrm{M_t/M/1/\infty}$ queue is a random variable having, among other things, a finite expectation. In particular, one can take the partial derivative with respect to x in (3.9) at $x = y = 1$. Doing this and using (3.10), we obtain the following first order differential equation

$$\tau\frac{d\alpha(t)}{dt} + \alpha(t)\big(\mu\tau + 1 - F(1,0,t)\big) = \mu U. \quad (3.11)$$

For the sake of shortness, set $U(y,t) \stackrel{\text{def}}{=} F(1,y,t)$. Instantiating $x = 1$ in (3.9) yields now

$$\frac{\partial U(y,t)}{\partial t} + \left(\lambda(1-y) + \alpha(t)\Big(1-\frac{1}{y}\Big)\right)U(y,t) = \alpha(t)\Big(1-\frac{1}{y}\Big)U(0,t), \quad (3.12)$$

which is nothing else but the evolution equation of a $M/M_t/1/\infty$ queue, i.e. a birth and death process with time-dependent parameters. A direct integration of (3.12) with respect to t leads to

$$\varphi(y,t)U(y,t) = U(y,0) + \Big(1-\frac{1}{y}\Big)\int_0^t \varphi(y,s)\alpha(s)U(0,s)ds\,, \tag{3.13}$$

where

$$\varphi(y,t) = \exp\Big\{\Big(1-\frac{1}{y}\Big)\int_0^t (\alpha(s)-\lambda y)ds\Big\}.$$

But $U(y,t)$, from its very definition, has to be holomorphic in the region $|y|<1$ and Cauchy's formula applies. In particular

$$H(t) \stackrel{\text{def}}{=} U(0,t) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{U(z,t)}{z}dz, \tag{3.14}$$

where $\mathcal{L}$ is a simple closed contour around 0 and contained in the unit disk. Combining (3.13) and (3.14), one gets for $H(t)$ the following Volterra integral equation of the second kind

$$H(t) = \psi(t) + \int_0^t \alpha(s)D(s,t)H(s)ds\,, \tag{3.15}$$

with

$$\psi(t) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{U(z,0)}{z\varphi(z,t)}dz\,, \qquad D(s,t) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(z,s)}{\varphi(z,t)}\frac{1-\frac{1}{z}}{z}dz.$$

It turns out that $D(s,t)$ can be expressed in terms of the modified Bessel functions

$$I_n(z) = \frac{1}{2i\pi}\int_{|\omega|=1} \frac{e^{\frac{z}{2}(\omega+\frac{1}{\omega})}}{\omega^{n+1}}d\omega, \quad n\in\mathcal{Z}.$$

Setting $A_t \stackrel{\text{def}}{=} \int_0^t \alpha(s)ds$, we have exactly

$$\begin{aligned} D(s,t) = e^{-\lambda(t-s)-(A_t-A_s)}\Big[I_0\Big(2\sqrt{\lambda(t-s)(A_t-A_s)}\Big) \\ - \Big(\frac{\lambda(t-s)}{A_t-A_s}\Big)^{1/2} I_1\Big(2\sqrt{\lambda(t-s)(A_t-A_s)}\Big)\Big]. \end{aligned} \tag{3.16}$$

To summarize, $\alpha(t)$ and $H(t)$ satisfy the system of integro-differential equations (3.11) and (3.15), which will be shown to have a unique solution in the class $\mathcal{C}[0,\infty]$.

In fact in (3.15), ψ and D are functionals of $\alpha(t)$. Therefore, in the sequel and in particular in the forthcoming lemma, to emphasize the functional dependence on $\alpha(t)$, one will add the subscript "α" to the concerned functions, e.g. $H_\alpha(t), \psi_\alpha(t), D_\alpha(s,t)$. With this convention, the above system rewrites

$$\begin{cases} \tau \dfrac{d\alpha(t)}{dt} + \alpha(t)\big(\mu\tau + 1 - H_\alpha(t)\big) = \mu U, \\ H_\alpha(t) = \psi_\alpha(t) + \int_0^t \alpha(s) D_\alpha(s,t) H_\alpha(s) ds. \end{cases} \tag{3.17}$$

The essence of theorem 2.2 is contained in the next lemma.

Lemma 3.1 *For any given initial condition $\alpha(0) \in [0, U/\tau]$, the differential system (3.17) has a unique solution and*

$$\lim_{t\to\infty} \alpha(t) = \begin{cases} \dfrac{U-\rho}{\tau}, & \textit{if} \quad \rho + \lambda\tau < U, \\ \dfrac{\mu U}{1+\mu\tau}, & \textit{otherwise.} \end{cases} \tag{3.18}$$

Moreover, in the ergodic case the convergence is exponential, and we have more precisely

$$\alpha(t) = \frac{U-\rho}{\tau} + \mathcal{O}(e^{-vt}), \tag{3.19}$$

where

$$v = \min\left[\frac{\lambda}{\tau\alpha} + \mu, \left(\sqrt{\alpha} - \sqrt{\lambda}\right)^2\right]$$

can be thought of the relaxation time *of the system.*

Proof Since $0 \le H_\alpha(t) \le 1$, integrating the first equation of (3.17) yields the bounds

$$\frac{\mu U}{1+\mu\tau}\left[1 - \exp\left(\frac{-\mu U t}{1+\mu\tau}\right)\right] \le \alpha(t) \le \frac{U}{\tau},$$

which are consistant with (2.8). We introduce now the following basic iterative scheme.

$$
\begin{cases}
\alpha_0(t) = \dfrac{U}{\tau}, \quad \forall t \geq 0\,, \\
\tau \dfrac{d\alpha_{n+1}(t)}{dt} + \alpha_{n+1}(t)\big(\mu\tau + 1 - H_{\alpha_n}(t)\big) = \mu U, \\
\alpha_{n+1}(0) \equiv \alpha(0) \in [0, U/\tau], \\
H_{\alpha_n}(t) = \psi_{\alpha_n}(t) + \int_0^t \alpha_n(s) D_{\alpha_n}(s,t) H_{\alpha_n}(s) ds, \quad n \geq 0.
\end{cases}
\tag{3.20}
$$

For each $n \geq 0$, let $\mathcal{Q}_n$ denote the $\mathrm{M/M_t/1/\infty}$ queue having $\alpha_n(\cdot)$ as service rate funtion. The probability for $\mathcal{Q}_n$ to be empty at time t is exactly equal to $H_{\alpha_n}(t)$, the unique solution of the Volterra integral equation appearing in system (3.20). Starting from some simple stochastic monotonicity properties proved in Appendix A.1, we shall argue by induction on n, assuming that all $\mathcal{Q}_n$'s have the same initial conditions [it is worth emphasizing that this argument is rendered possible by the third equation in (3.20)].

Suppose $\alpha_n(t) \leq \alpha_{n-1}(t)$, which is in particular true for $n = 1$. Consequently, by property (ii) of lemma A.1, $H_{\alpha_n}(t) \leq H_{\alpha_{n-1}}(t)$. Letting

$$
L_n(t) \stackrel{\text{def}}{=} \int_0^t \Big(\frac{1 - H_{\alpha_n}(s)}{\tau} + \mu\Big) ds,
$$

the second equation of (3.20) implies

$$
\alpha_{n+1}(t) = \exp\big(-L_n(t)\big)\Big[\alpha(0) + \frac{\mu U}{\tau}\int_0^t \exp(L_n(s)) ds\Big]. \tag{3.21}
$$

Moreover, the form of (3.21) shows at once that $\alpha_{n+1}(t)$ is an increasing functional of H_{α_n}, so that

$$
\alpha_{n+1}(t) \leq \alpha_n(t) \leq \frac{U}{\tau}.
$$

Thus the sequences $\{\alpha_n(t), n \geq 0\}$ and $\{H_{\alpha_n}(t), n \geq 0\}$ are for each fixed t non-increasing and uniformly bounded. Consequently, the functions

$$
\alpha(t) = \lim_{n\to\infty} \alpha_n(t) \quad \text{and} \quad H_\alpha(t) = \lim_{n\to\infty} H_{\alpha_n}(t)
$$

do exist and satisfy (3.17).
Still operating by induction on n in (3.21), suppose for some $n > 1$ the existence of the quantity

$$\alpha_n \stackrel{\text{def}}{=} \lim_{t\to\infty} \alpha_n(t)\,,$$

and note that this statement holds for $n = 0$. Then a coupling argument based on the aforementioned stochastic monotonicity yields directly

$$\lim_{t\to\infty} H_{\alpha_n}(t) = \max\left(0, 1 - \frac{\lambda}{\alpha_n}\right).$$

Upon applying now lemma B.1 to (3.21), we obtain

$$\lim_{t\to\infty} \alpha_{n+1}(t) = \lim_{t\to\infty} \frac{\mu U}{\tau} \int_0^t \exp(L_n(s) - L_n(t))ds = \frac{\mu U}{\mu\tau + \min(1, \lambda/\alpha_n)},$$

which concludes the main step of the induction. We can write

$$\begin{cases} \alpha = \lim\limits_{t\to\infty} \alpha(t) = \lim\limits_{t\to\infty}\lim\limits_{n\to\infty} \alpha_n(t) = \lim\limits_{n\to\infty}\lim\limits_{t\to\infty} \alpha_n(t), \\ \lim\limits_{t\to\infty} H_\alpha(t) = \lim\limits_{t\to\infty}\lim\limits_{n\to\infty} H_{\alpha_n}(t) = \max\left(0, 1 - \dfrac{\lambda}{\alpha}\right), \end{cases} \tag{3.22}$$

where the permutation of limits in (3.22), which corresponds in some sense to the uniform convergence of the sequence $\{\alpha_n(t)\}$, is ensured by the following argument. Indeed, it is easy to check that the new scheme

$$\begin{cases} \beta_0(t) = 0, \quad \forall t \geq 0\,, \\ \tau \dfrac{d\beta_{n+1}(t)}{dt} + \beta_{n+1}(t)\big(\mu\tau + 1 - H_{\beta_n}(t)\big) = \mu U, \\ \beta_{n+1}(0) \equiv \alpha(0) \in [0, U/\tau], \\ H_{\beta_n}(t) = \psi_{\beta_n}(t) + \int_0^t \beta_n(s) D_{\beta_n}(s,t) H_{\beta_n}(s) ds, \quad n \geq 0, \end{cases} \tag{3.23}$$

(which differs from (3.20) only by its first equation) is monotone increasing, and that

$$\lim_{n\to\infty} \beta_n(t) = \alpha(t), \quad \text{with} \quad \beta_n(t) \leq \alpha(t) \leq \alpha_n(t), \quad \forall t, n \geq 0.$$

Letting first $t \to \infty$,

$$\beta_n \le \liminf \alpha(t) \le \limsup \alpha(t) \le \alpha_n,$$

then $n \to \infty$, we obtain the desired property.

Finally, since $\alpha(t)$ is uniformly bounded, $\lim_{t\to\infty} \dfrac{d\alpha(t)}{dt} = 0$, and hence, from the first equation in (3.17),

$$\alpha = \frac{\mu U}{\mu\tau + \min(1, \lambda/\alpha)},$$

which is tantamount to (3.18). The result contained in equation (3.19) concerns the speed of convergence and is proved in Appendix B.

After having shown existence of a solution of system (3.17), we are left with the problem of uniqueness. In fact, considering the first equation of (3.17) and using standard results on integral equations, the uniqueness is here straightforward, since by (iii) of lemma A.1 $H_\alpha(\cdot)$ satisfies a Lipschitz condition with respect to $\alpha(t)$. This concludes the proof of the lemma together with STEP A of the theorem. ∎

3.2.2 STEP B

From the preceding argument, we are entitled now to analyze the forward equations (2.7), by handling $\alpha(t)$ as if it were an *exogenous* given function, solution of (3.17), subject to the additional initial condition, derived from (2.8),

$$\alpha(0) = \frac{U}{\tau}\Big(1 - \frac{1}{U}\sum_{j,k\ge 0} j\alpha_{jk}(0)\Big).$$

Then, since we are facing birth and death processes, existence and uniqueness of a solution to the forward Kolmogorov's equations (2.7) follow at once from the general theory (see [7, 10]). In fact, in each compact set and for all $t < \infty$, the generator of the process $\xi(t)$ which represents the tandem queue network in 3.2 is bounded, and hence equations (2.7) have a unique solution satisfying

$$\sum_{j,k} \alpha_{j,k}(t) \le 1. \tag{3.24}$$

Introducing θ_k, the k-th jump of $\xi(t)$, we have $|\xi(\theta_k)| \leq |\xi(0)|+k$ and, given the past, $\theta_{k+1}-\theta_k$ has an exponential distribution with mean at least A/k, A being a positive constant. Thus $\xi(t)$ is regular, since $\sum 1/k = \infty$, and in particular it has only finitely many jumps in any finite time interval. Consequently, there is equality in (3.24), and (2.7) has a unique probabilistic solution.

The final question is the analysis of the stationary distribution of the two-dimensional random walk $\xi(t)$, when $t \to \infty$. Clearly, the expected result is that this measure should exist and coincide with the invariant measure of the random walk, say $\widetilde{\xi}(t)$, obtained just replacing the function $\alpha(t)$ by the constant α. This will prove (2.10) and (2.11), since $\widetilde{\xi}(t)$ corresponds to a standard Jackson network, known to have a product form at equilibrium. In fact, the result is true and can be derived in two different ways.

Coupling. It turns out that $\xi(t)$ and $\widetilde{\xi}(t)$ can be coupled in an easy way. Indeed, for all $\epsilon > 0$, there exists T_ϵ such that

$$|\alpha(t) - \alpha| < \epsilon, \quad \forall t \geq T_\epsilon.$$

Taking now $\widetilde{\xi}(0) = \xi(T_\epsilon)$, the result follows directly from inequality (A.5) and sections A.1–A.3. Remark that the output process of the M/M/1/∞ queue tends in law, as $t \to \infty$, to a Poisson process with intensity λ, so that the second queue in the tandem network $\xi(t)$ tends to a M/M/∞ infinite server queue, with parameters α and μ. Note that the result holds without any prerequisite about the speed of convergence of $\alpha(t)$ to its limit α.

Analytical approach. As a mere luxury, we tackle here the problem from an angle which should concern analytically-minded readers.

Denote by $\widetilde{F}(x,y,t)$ the generating function corresponding to $\widetilde{\xi}(t)$, which satisfies a functional equation analogous to (3.18), just replacing $\alpha(t)$ by α. The initial conditions in boths systems are supposed to be identical.

Setting $E(x,y,t) \stackrel{\text{def}}{=} \widetilde{F}(x,y,t) - F(x,y,t)$, it is easy to check that E satisfies the following functional equation:

$$\frac{\partial E(x,y,t)}{\partial t} + \left(\lambda(1-y) + \alpha\left(1-\frac{x}{y}\right)\right)E(x,y,t) = \mu(1-x)\frac{\partial E(x,y,t)}{\partial x}$$
$$+\,\alpha\left(1-\frac{x}{y}\right)E(x,0,t) + \left(1-\frac{x}{y}\right)(\alpha(t)-\alpha)\big(F(x,y,t)-F(x,0,t)\big). \quad (3.25)$$

The main lines of argument to show that

$$\lim_{t\to\infty} \frac{\partial E(x,y,t)}{\partial t} = 0,$$

are presented in Appendix A.3. The proof of STEP B is terminated. ∎

3.3 Proof of theorem 2.3

This section needs notions and results which are presented in Appendix C. To begin with, we characterize the limiting operator $G = A + B$ introduced in equations (2.5) and (2.6). The results are basically contained in the forthcoming lemma.

Lemma 3.2 *$\overline{G}$, the closure of G, generates a strongly continuous positive contraction semigroup $\{T(t),\ t \geq 0\}$ on $\mathcal{C}_0$ and $\mathcal{C}_c \cap \mathcal{C}_1$ is a core for $\overline{G}$*

Proof At once, it is worth noting that A (resp. B) leaves $\vec{\alpha}$ (resp. the pair $(\vec{q}, \vec{x})$) invariant. The fact that $\overline{B}$ is a generator follows from (2.9) and the related remarks. As far as A is concerned, most of the techniques used below are borrowed from [6].

Let $\gamma(\vec{x}) \stackrel{\text{def}}{=} \sum_{i=1}^{r} x_i$. Then, for all $f \in \mathcal{C}_0$ such that $\|\gamma f\| < \infty$, the operator A generates a strongly continuous positive contraction semigroup on $\mathcal{C}_0$, with domain $\mathcal{D}(A)$ (see [6], section 8.3). Since A is not bounded, the simplest way to show that $\overline{G}$ is also the generator of a strongly positive continuous contraction semigroup consists in checking the three conditions of the Hille-Yosida theorem C.1 stated in Appendix C.

For any $f \in \mathcal{D}(G) = \mathcal{D}(A) \cap \mathcal{D}(B)$, we have

$$Gf(\vec{q}, \vec{x}, \vec{\alpha}) = Af(\vec{q}, \vec{x}, \vec{\alpha}) + Bf(\vec{q}, \vec{x}, \vec{\alpha}).$$

First, it is clear that the domain $\mathcal{D}(G)$ is dense in $\mathcal{C}_0$. Secondly, as in [6], p. 37, one can show that G is dissipative. The last point is to verify that $\mathcal{R}(\beta - G)$ is dense in $\mathcal{C}_0$, for some positive β. To this end, we will use perturbation methods.

Let, for $f \in \mathcal{C}_0$ and each $n > 0$,

$$
\begin{aligned}
A_n f(\vec{q},\vec{x},\vec{\alpha}) &\stackrel{\text{def}}{=} \lambda \sum_{i=1}^{r} \big(f(\vec{q}+\vec{e}_i,\vec{x},\vec{\alpha}) - f(\vec{q},\vec{x},\vec{\alpha}) \big) \\
&+ \mu \frac{\gamma(\vec{x}) \wedge n}{\gamma(\vec{x})} \sum_{i=1}^{r} x_i \big(f(\vec{q}-\vec{e}_i,\vec{x}-\vec{e}_i,\vec{\alpha}) - f(\vec{q},\vec{x},\vec{\alpha}) \big) \\
&+ \frac{Ua}{\tau} \sum_{i=1}^{r} \mathbb{1}_{\{x_i<q_i\}} (f(\vec{q},\vec{x}+\vec{e}_i,\vec{\alpha}) - f(\vec{q},\vec{x},\vec{\alpha})),
\end{aligned}
$$

where

$$
a = 1 - \frac{1}{U} \sum_{j,k \ge 0} j\alpha_{jk}.
$$

Since A_n is bounded, $G_n \stackrel{\text{def}}{=} A_n + \overline{B}$ generates a strongly continuous positive contraction semigroup $\{T_n(t),\, t \ge 0\}$ on $\mathcal{C}_0$. From the definition of γ, there exists $\omega \ge 0$ not depending on n such that $\|\gamma G_n(1/\gamma)\| \le \omega$. Hence,

$$
e^{-\omega t} T_n(t)\frac{1}{\gamma} - \frac{1}{\gamma} = \int_0^t e^{-\omega s} T_n(s) \left(G_n \frac{1}{\gamma} - \frac{\omega}{\gamma} \right) ds \le 0,
$$

for all $t \ge 0$ and any $g \in \mathcal{D}(B)$ with $\|\gamma g\| < \infty$. Consequently

$$
\|\gamma T_n(t) g\| \le e^{\omega t} \|\gamma g\|,
$$

and $T_n(t)$ is one-to-one from $\mathcal{C}_1$ to $\mathcal{C}_1$. Setting now, for any $\beta > \omega$,

$$
f_{m,n} = \frac{1}{m} \sum_{k=0}^{m^2} e^{-\frac{\beta k}{m}} T_n\Big(\frac{k}{m}\Big) g, \quad \forall m, n \ge 0,
$$

it is clear that $f_{m,n} \in \mathcal{D}(G)$, since $f_{m,n} \in \mathcal{C}_1$ and $\|\gamma f_{m,n}\|$ is uniformly bounded. Using standard properties of $\{T_n(t)\}$, we find that

$$
(\beta - G_n) f_{m,n} = \frac{1}{m} \sum_{k=0}^{m^2} e^{-\frac{\beta k}{m}} T_n\Big(\frac{k}{m}\Big) (\beta - G) g, \quad \forall m, n \ge 0,
$$

which in turn implies

$$\lim_{m\to\infty} \|(\beta - G_n)f_{m,n} - g\| = 0, \quad \forall n \geq 0.$$

Out of $\{f_{m,n}\}$, one can extract a subsequence $\{f_n, n \geq 0\}$ such that

$$\|(\beta - G_n)f_n - g\| \leq \epsilon_n, \quad \text{with } \lim_{n\to\infty} \epsilon_n = 0. \tag{3.26}$$

Then

$$(\beta - G)f_n - g = (\beta - G_n)f_n - g + (A_n - A)f_n,$$

where

$$(A_n - A)f_n(\vec{q}, \vec{x}, \vec{\alpha}) =$$
$$\mu \frac{\min\big(n - \gamma(\vec{x}), 0\big)}{\gamma(\vec{x})} \sum_{i=1}^{r} x_i \big(f_n(\vec{q} - \vec{e}_i, \vec{x} - \vec{e}_i, \vec{\alpha}) - f_n(\vec{q}, \vec{x}, \vec{\alpha})\big).$$

By (3.26) and the fact that $\|\gamma f_n\|$ is uniformly bounded, the dominated convergence theorem shows that

$$\lim_{n\to\infty} \int \Big((\beta - G)f_n - g\Big) d\varphi = 0,$$

where φ is an arbitrary Radon measure. Thus $\mathcal{R}(\beta - G)$ is weakly dense in $\mathcal{C}_0$, and hence also dense in $\mathcal{C}_0$. Then $\overline{G}$ generates a strongly continuous positive contraction semigroup and the subspace of functions

$$\mathcal{D} \stackrel{\text{def}}{=} \{f \in \mathcal{C}_1, \|\gamma f\| < \infty\}$$

is a core for $\overline{G}$. To show that $\mathcal{C}_c \cap \mathcal{C}_1$ is also a core for $\overline{G}$, take $g \in \mathcal{D}$ and consider the sequence $\{g_n\} \subset \mathcal{C}_c \cap \mathcal{C}_1$ where

$$g_n = g \mathbb{1}_{\{q_i \leq n, x_i \leq n, \forall i\}}, \quad \forall n \geq 0.$$

Then it is not difficult to see that

$$\lim_{n\to\infty} \int \Big(Gg_n - Gg\Big) d\varphi = 0,$$

where φ is an arbitrary Radon measure. Lemma 3.2 is proved. ■

Now, theorem 2.3 is a consequence of proposition C.4, as follows.
Let $\varphi_N : \Lambda^{(N,r)} \to \Lambda$ with

$$\varphi_N(\vec{\alpha}) = \Big(1 - \frac{U(N-r)}{V^{(N)}}\Big)\vec{e}_{00} + \frac{U(N-r)}{V^{(N)}}\vec{\alpha}\ , \quad \vec{\alpha} \in \Lambda^{(N,r)}.$$

We have

$$\varphi_N(\vec{\alpha}+\vec{\beta}) = \varphi_N(\vec{\alpha}) + \frac{U(N-r)}{V^{(N)}}\ \vec{\beta}\ , \quad \vec{\alpha},\ \vec{\beta} \in \Lambda^{(N,r)}.$$

Let Π_N be the bounded linear mapping from $\mathcal{L}$ into $\mathcal{L}_N$ defined by

$$\Pi_N f(\vec{q},\vec{x},\vec{\alpha}) = f(\vec{q},\vec{x},\varphi_N(\vec{\alpha})), \quad \forall(\vec{q},\vec{x},\vec{\alpha}) \in \mathcal{E}^r \otimes \Lambda^{(N,r)}.$$

Now, by property (c) of theorem C.4, it suffices to check that

$$\lim_{N\to\infty} \|G^{(N,r)}\Pi_N f - \Pi_N G f\| = 0, \quad f \in \mathcal{C}_c \cap \mathcal{C}_1. \tag{3.27}$$

But, since $f \in \mathcal{C}_c \cap \mathcal{C}_1$ has a compact support and $\lim_{N\to\infty} V^{(N)}/N = U$, equation (3.27) follows simply by applying Taylor's formula in $G^{(N,r)}$ defined by (2.4). This concludes the proof of theorem 2.3. ■

3.4 Proof of theorem 2.4

This theorem is a straight consequence of the tightness (proved in section 3.1) of the family of stationary measures associated to the processes $\vec{Y}^{(N,r)}$, and of the following theorem ([6], theorem 9.10 p. 244).

Theorem 3.3 *Let $\{T_n(t), n > 0\}, \{T(t)\}$ be contraction semigroups corresponding to Markov processes in a complete separable metric space E, and suppose that for each n, μ_n is a stationary distribution for $\{T_n(t), n > 0\}$. Let $L \subset \mathcal{C}(E)$ be separating for the weak convergence and $T(t) : L \to\ L$ for all $t \geq 0$. Suppose that for each $f \in L$ and compact $K \subset E$,*

$$\lim_{n\to\infty} \sup_{x\in K} |T_n(t)f(x) - T(t)f(x)| = 0, \quad t \geq 0.$$

Then every weak limit point of $\{\mu_n\}$ is a stationary distribution for $\{T(t)\}$.

3.5 Proof of theorem 2.5

The initial state $\vec{Z}^{(N,r)}(0)$ of the system $\mathcal{P}^{(N)}$ is supposed now to be arbitrary, but finite. This means in particular that, at time $t = 0$, the vehicles are randomly distributed in the network, so that they a priori do not choose their destinatation with probability $1/N$, until they have visited a node. But, if the initial backlog of customers is uniformly bounded with respect to $N \geq 0$ (this case covers all but degenerate situations), then we know, under conditions (2.1) and (2.2), that the mean stationary number of waiting customers at each node remains also uniformly bounded with respect to N, no matter where the servers are originally located (see the computation of the variable D in section 3.1). Thus there exists a finite random time $T^{(N)}$, after which $\mathcal{P}^{(N)}$ will evolve according to the generators $G^{(N)}$ and $G^{(N,r)}$, since all servers will have visited each node at least once. The proof is terminated.

3.6 About theorems 2.6, 2.7, 2.8 and conclusion

The reader understands that all theorems concerning block-wise symmetrical systems could be obtained along the same lines as in the symmetrical case, at the expense of a heavier notation, without new ideas. Therefore they will be omitted. Let us mention that the ergodicity conditions, quoted in theorem 2.6, can be obtained by combining the construction of linear Lyapounov functions with the argument of section 3.1. Also, from equations (2.13) and (2.14), one can get the following system of integro-differential equations, which does not differ fundamentally from (3.17), apart from its dimension:

$$\begin{cases} J_r(t) = \psi_r(t) + \int_0^t \eta_r(u) D_r(u,t) J_r(u) du, \\ \dfrac{dM_r(t)}{dt} + \mu_r J_r(t) = \eta_r(t)(1 - J_r(t)), \\ \tau_{rs} \dfrac{d\eta_{rs}(t)}{dt} + \eta_{rs}(t) = \dfrac{p_{rs} U_s}{U_r} \big[\mu_r M_r(t) + \eta_r(t) J_r(t)\big], \quad \forall r, s \in \mathcal{K}. \end{cases} \tag{3.28}$$

Essentially as in section 3.2, one can show that system (3.28) has a unique solution. Obviously more serious difficulties arise in allowing the number of blocks, say K_N, to be a function of N, possibly growing to infinity with N.

Nevertheless, one can safely conjecture that theorems 2.7 and 2.8 of section 2.2 remain valid under natural assumptions about the limiting routing matrix between blocks: the dynamical system must now be viewed as an *infinite* number of tandem queues in parallel, each tandem having the structure shown in figure 3.2. This could be a basic step forward to quantify the propagation of chaos in general polling systems.

A Queues with time-dependent parameters

A.1 The $\mathrm{M}/\mathrm{M}_{\mathrm{t}}/1/\infty$ queue

This section briefly presents some basic features of the operator describing the evolution of the $\mathrm{M}/\mathrm{M}_{\mathrm{t}}/1/\infty$ queue, with constant arrival rate λ and time varying service rate $\beta(\cdot) \in \mathcal{C}^+[0,\infty]$.

The probabilities $p_n(t) \stackrel{\text{def}}{=} P(Z(t)=n)$, $n \geq 0$, where Z(t) denotes the number in the queue at time t, obey the following set of *forward* differential equations:

$$\begin{cases} \dfrac{dp_0(t)}{dt} = -\lambda p_0(t) + \beta(t) p_1(t), \\ \dfrac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - (\lambda + \beta(t)) p_n(t) + \beta(t) p_{n+1}(t), \quad n \geq 1, \end{cases} \tag{A.1}$$

which we shall rewrite in operator form

$$\frac{d\mathbf{P}_\beta(t)}{dt} = \mathbf{P}_\beta(t)\mathbf{K}_\beta(t), \tag{A.2}$$

where $\mathbf{K}_\beta(t)$ is a generator (represented by an infinite matrix) and $\mathbf{P}_\beta(t)$ is an infinite row vector belonging to the Banach space ℓ_1 of absolutely summable sequences. It is known either from a probabilistic point of view (e.g. [7, 10]) or by an analytic argument (e.g. [4]), that (A.2) has, for all $t \geq 0$, a unique solution in ℓ_1. In addition, the generator $\mathbf{K}_\beta(t)$ has a continuous spectrum of eigenvalues, located on the negative real line.

Similarly, the distribution function

$$s_n(t) \stackrel{\text{def}}{=} P(Z(t) \leq n), \quad \forall n \geq 0,$$

satisfies the system

$$\begin{cases} \dfrac{ds_0(t)}{dt} = -(\lambda+\beta(t))s_0(t)+\beta(t)s_1(t), \\ \dfrac{ds_n(t)}{dt} = \lambda s_{n-1}(t)-(\lambda+\beta(t))s_n(t)+\beta(t)s_{n+1}(t), \quad n\geq 1, \end{cases} \tag{A.3}$$

which will be written as

$$\frac{d\mathbf{S}_\beta(t)}{dt} = \mathbf{S}_\beta(t)\mathbf{L}_\beta(t), \tag{A.4}$$

where $\mathbf{S}_\beta(t)$ denotes the row vector $\mathbf{S}_\beta(t) \stackrel{\text{def}}{=} (s_0(t), s_1(t), \dots)$.

Lemma A.1 *Let $\mathbf{P}_\beta(t)$ and $\widetilde{\mathbf{P}}_\beta(t)$ be the solutions of (A.2) corresponding to respective initial conditions $\mathbf{P}_\beta(0)$ and $\widetilde{\mathbf{P}}_\beta(0)$. The following properties hold:*

(i) *If $\mathbf{P}_\beta(0) \geq \widetilde{\mathbf{P}}_\beta(0) \geq 0$, then $\mathbf{P}_\beta(t) \geq \widetilde{\mathbf{P}}_\beta(t) \geq 0,\ \forall t\geq 0$.*

(ii) *Let $\beta(\cdot),\gamma(\cdot) \in \mathcal{C}^+[0,\infty]$, such that*

$$\beta(t)\leq\gamma(t), \quad \forall t\geq 0, \quad \textit{and} \quad \mathbf{S}_\beta(0)\leq\mathbf{S}_\gamma(0).$$

Then (stochastic dominance)

$$\mathbf{S}_\beta(t)\leq\mathbf{S}_\gamma(t),\ \forall t\geq 0.$$

In particular, with the notation of equation (3.17), $H_\beta(t)\leq H_\gamma(t), \forall t$.

(iii) *For any $\beta(\cdot),\gamma(\cdot)\in\mathcal{C}^+[0,\infty]$, with $\mathbf{P}_\beta(0)=\mathbf{P}_\gamma(0)$, we have* the Lipschitz condition

$$|\mathbf{P}_\beta(t)-\mathbf{P}_\gamma(t)| \leq K\|\beta(\cdot)-\gamma(\cdot)\|, \tag{A.5}$$

where K is an absolute constant independent of t and $|\cdot|$ denotes, for an arbitrary vector X, the usual ℓ_1-norm $|X|=\sum_i |x_i|$.

(iv) *The* $\mathrm{M/M_t/1/\infty}$ *queue with service rate $\alpha(t)$ and the* $\mathrm{M/M/1/\infty}$ *queue with service rate $\alpha=\lim_{t\to\infty}\alpha(t)$ have the same stationary regime.*

Proof It is not difficult to see that $\mathbf{K}_\beta(t)$ and $\mathbf{L}_\beta(t)$ are positive operators. For instance, making in A.3 the change of functions

$$s_n(t) = w_n(t) \exp\Big(-\lambda t - \int_0^t \beta(s)ds\Big), \quad n \geq 0,$$

leads to the system

$$\begin{cases} \dfrac{dw_0(t)}{dt} = \beta(t) w_1(t), \\ \dfrac{dw_n(t)}{dt} = \lambda w_{n-1}(t) + \beta(t) w_{n+1}(t), \quad n \geq 1, \end{cases}$$

which has the form

$$\frac{d\mathbf{W}_\beta(t)}{dt} = \mathbf{W}_\beta(t) \widetilde{\mathbf{L}}_\beta(t),$$

where $\widetilde{\mathbf{L}}_\beta(t)$ has only positive coefficients. A similar argument can be used for the positivity of $\mathbf{K}_\beta(t)$.

The stochastic dominance in (ii) follows now from (i). Indeed, setting

$$\mathbf{R}(t) = \mathbf{S}_\gamma(t) - \mathbf{S}_\beta(t),$$

the vector $\mathbf{R}(t)$ satifies the non homogeneous differential equation

$$\frac{d\mathbf{R}(t)}{dt} = \mathbf{R}(t)\mathbf{L}_\gamma(t) + \big(\gamma(t) - \beta(t)\big)\mathbf{D}(t), \tag{A.6}$$

where $\mathbf{D}(t) = (d_0(t), d_1(t), \dots)$, with $d_n(t) = [s_{n+1}(t) - s_n(t)]_\beta$. By (i), the vector $\mathbf{D}(t)$ has non-negative components and the operator $\mathbf{L}_\gamma(t)$ is positive, whence it follows that the solutions of (A.6) are also non negative.

To prove (iii), we shall use differential calculus in Banach spaces. In this framework, most of the classical results for the real line or the complex plane apply without substantial modification.

For any $\beta(\cdot) \in \mathcal{C}^+[0,\infty]$, with $0 \leq \|\beta\| \leq U/\tau$, take an arbitrary perturbation function $\Delta(\cdot)$, with $\beta(t) + \Delta(t) \in \mathcal{C}^+[0,\infty]$. When it exists, the partial derivative with respect to $\beta(\cdot)$ of a differentiable mapping

$$g : \mathcal{C}^+[0,\infty] \times [0,\infty] \to \ell_1$$

is a functional (see [4]) written $\frac{\partial g(t)}{\partial \beta}$. With this notation, one sees easily that $\mathbf{Q}_\beta \stackrel{\text{def}}{=} \frac{\partial \mathbf{P}_\beta(t)}{\partial \beta}$, where P_β satisfies (A.2), must be solution of the following non-homogeneous linear differential equation

$$\frac{d\mathbf{Q}_\beta(t)}{dt} = \mathbf{Q}_\beta(t)\mathbf{K}_\beta(t) + \mathbf{P}_\beta(t)\mathbf{M}, \quad \mathbf{Q}(0) = 0, \tag{A.7}$$

where $\mathbf{M}$ is a constant infinite matrix given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}.$$

The solution of (A.7) writes in the form

$$\mathbf{Q}_\beta(t) = \int_0^t \mathbf{P}_\beta(s)\mathbf{M}\Phi(t,s)ds,$$

where $\Phi(t,s)$ is the so-called fundamental solution (see [4]) of the homogeneous equation of type (A.2). The preceding argument yields directly the rough estimate

$$\sup_\beta \|\mathbf{Q}_\beta\| = \sup_\beta \sup_{t\geq 0} \|\mathbf{Q}_\beta(t)\| \leq K, \tag{A.8}$$

where K is a bounded constant. Since $\mathbf{Q}_\beta$ is the derivative with respect to $\beta(\cdot)$ of the function $\mathbf{P}_\beta$, defined on the Banach space $\mathcal{C}^+[0,\infty]$, (A.8) gives the Lipschitz condition (A.5).

Although the last property in the lemma could be viewed as an immediate consequence of the stochastic ordering contained in (i) and (ii), we think it might be also interesting to derive it analytically. Setting $V(y,t) \stackrel{\text{def}}{=} E(1,y,t)$ in (3.25), $V(y,t)$ satisfies the equation

$$\frac{\partial V(y,t)}{\partial t} + \left(\lambda(1-y) + \alpha\Big(1-\frac{1}{y}\Big)\right)V(y,t) = \\ + \alpha\Big(1-\frac{1}{y}\Big)V(0,t) + \Big(1-\frac{1}{y}\Big)(\alpha(t)-\alpha)\theta(y,t), \tag{A.9}$$

where $\theta(y,t) = F(1,y,t) - F(1,0,t)$. For our purpose, it suffices to show that

$$\lim_{t\to\infty} \frac{\partial V(y,t)}{\partial t} = 0, \quad \forall\, |y| \le 1.$$

Taking ordinary Laplace transform in both members of (A.9), we obtain, since $V(y,0) = 0$,

$$\left(s + \lambda(1-y) + \alpha\Big(1 - \frac{1}{y}\Big)\right) V^*(y,s) = \Big(1 - \frac{1}{y}\Big)\big(\alpha V^*(0,s) - \psi^*(y,s)\big), \tag{A.10}$$

for $\Re(s) \ge 0$, with the usual notation and

$$\psi^*(y,s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st}(\alpha - \alpha(t))\theta(y,t)dt.$$

If $y(s)$ denotes the unique root of the equation

$$s + \lambda(1-y) + \alpha\Big(1 - \frac{1}{y}\Big) = 0,$$

satisfying $|y(s)| \le 1$, for $\Re(s) \ge 0$, then it follows readily from (A.10) that

$$\alpha V^*(0,s) = \psi^*(y(s), s),$$

or, equivalently,

$$\alpha \int_0^\infty e^{-st} V(0,t)dt = \int_0^\infty e^{-st}(\alpha - \alpha(t))\theta(y(s),t)dt\,. \tag{A.11}$$

Noting that the functions $V(0,t)$ and $\theta(y,t)$ are uniformly bounded, as well as their derivatives with respect to t and y, for all $t \ge 0$, $|y| \le 1$, we are in a position to apply Tauber's theorem (or even directly the inverse Laplace transform) to the left-hand side member of (A.11). So, skipping straightforward computations which exploid the fact that $\lim_{t\to\infty} \alpha(t) = \alpha$, one can write

$$\lim_{t\to\infty} \frac{\partial V(0,t)}{\partial t} = \lim_{s\to 0} s\psi^*(y(s),s) = 0,$$

which implies property (iv) and concludes the proof of lemma A.1. ■

A.2 The $\mathrm{M_t/M}/\infty$ queue

Lemma A.2 *The* $\mathrm{M_t/M}/\infty$ *queue does enjoy properties which are quite similar to those listed in lemma A.1.*

The proof mimics that of lemma A.1, and details are therefore omitted. ∎

A.3 About the tandem queue $\mathrm{M/M_t}/\infty \to ./\mathrm{M}/\infty$ and functional equations of type (3.25)

Arguing again as in (i)–(iii) of lemma A.1, one easily obtains stochastic dominance properties for the queueing system drawn in figure 3.2, which in turn imply that the stationary regimes of the processes $\widetilde{\xi}(t)$ and $\xi(t)$ (introduced in section 3.2.2) are identical.
A direct analytical proof that

$$\lim_{t\to\infty} \frac{\partial E(x,y,t)}{\partial t} = 0, \tag{A.12}$$

where $E(x,y,t)$ is defined by (3.25), is in no way elementary. For the sake of reference, it is worth mentioning that the functional equation (3.25) to a boundary value problem, which is itself equivalent to a Fredholm integral equation for a function of one variable (see [5] for an equation of the same type). The analysis of this integral equation leads to (A.11), but the machinery is rather difficult and beyond the scope of the paper. In fact (A.12) can be derived by a more direct argument, sketched below, making use of semigroups on function spaces.
Let $\{S(t)\}$ and $\{\widetilde{S}(t)\}$ be the semigroups associated to $\xi(t)$ and $\widetilde{\xi}(t)$, noting that the corresponding generators B_t (which is not time-homogeneous) and $\widetilde{B}$ have the same domain, say D. Then the forward-type equation (3.25) has the equivalent operator form

$$\widetilde{S}(t)f = S(t)f + \int_0^t (\alpha(s)-\alpha)\widetilde{S}(t-s)\Gamma(s)f ds, \quad \forall f \in \mathcal{D}, \tag{A.13}$$

where $\Gamma(t)$ is in fact the difference of two linear positive bounded operators, representing the counterpart of $F(x,y,t)-F(x,0,t)$ in the right-hand side member of (3.25) [see a related form in equation (A.6)].

It suffices to show that the integral in (A.13) has a limit, as $t \to \infty$, which necessarily will be equal to 0. Rewrite, for any positive $g \in \mathcal{C}^+(E)$,

$$\int_0^t (\alpha(s)-\alpha)\widetilde{S}(t-s)g(s)ds = \\ \int_0^{t_0} (\alpha(s)-\alpha)\widetilde{S}(t-s)g(s)ds + \int_{t_0}^{t} (\alpha(s)-\alpha)\widetilde{S}(t-s)g(s)ds, \quad \text{(A.14)}$$

where, given $\epsilon > 0$, one chooses t_0 to ensure that $|\alpha(t)-\alpha| < \epsilon,\ \forall t \geq t_0$.
Since there exists $\lim_{t\to\infty} \widetilde{S}(t)$, which corresponds to the stationary distribution of the tandem queue with constant parameters, the first integral the right member of (A.14), has a limit. Moreover, from standard properties of the semigroup $\widetilde{S}(t)$, we have

$$\lim_{t\to\infty}\left|\int_{t_0}^{t} (\alpha(s)-\alpha)\widetilde{S}(t-s)g(s)ds\right| \leq K\epsilon,$$

where K is a positive constant. Thus the right member of (A.14) has a limit, as $t \to \infty$, and this gives another demonstration of assertion (A.12).

B Speed of convergence of $\alpha(t)$

We estimate, except for some exceptional values of the parameters, the main term in the asymptotic behaviour of $\alpha(t)$. In order to derive equation (3.19), the method we propose is to act again by induction, starting from the iterative scheme (3.20).

Lemma B.1 *Let F be a positive non-decreasing function defined on $\mathcal{R}^+$, with a continuous derivative f satisfying*

$$\lim_{t\to\infty} f(t) = q,$$

and define $I(t) \stackrel{def}{=} \int_0^t \exp(F(s)-F(t))ds$. Then

$$\lim_{t\to\infty} I(t) = \frac{1}{q}. \quad \text{(B.1)}$$

Supposes in addition $F(t) = qt + g(t)$, *where* $g(t)$ *is a uniformly differentiable function satisfying*

$$g(t) = bt^{-1/2}e^{-ct} + o\big(t^{-1/2}e^{-ct}\big), \quad as \quad t \to \infty,$$

b *and* c *being some constants, with* $c > 0$. *Then*

$$I(t) = \frac{1}{q} + \mathcal{O}\big(\max(e^{-qt}, t^{-1/2}e^{-ct})\big). \tag{B.2}$$

Proof We have

$$I(t) = I(s)\exp(F(s) - F(t)) + \int_s^t \exp(F(u) - F(t))du, \quad 0 \le s \le t. \tag{B.3}$$

Set $J(s,t) \stackrel{\text{def}}{=} \int_s^t \exp(F(u) - F(t))du$, and choose $\epsilon > 0$. From the assumption on F, there exists s_ϵ, such that

$$(u-t)(q+\epsilon) \le F(u) - F(t) \le (u-t)(q-\epsilon), \quad s_\epsilon \le u \le t.$$

Thus

$$\frac{1 - e^{-(t-s_\epsilon)(q+\epsilon)}}{q+\epsilon} \le J(s_\epsilon, t) \le \frac{1 - e^{-(t-s_\epsilon)(q-\epsilon)}}{q-\epsilon}. \tag{B.4}$$

As ϵ is arbitrary, (B.1) is obtained by letting $t \to \infty$ in (B.3) and (B.4).

The proof of (B.2) is more involved and we only sketch its main lines. Rewriting first $I(t)$ as

$$I(t) = J(t)\exp(-g(t))\,, \tag{B.5}$$

with

$$J(t) = \int_0^t \exp\big(-q(t-s) + g(s)\big)ds,$$

one sees $J(t)$ is a convolution and its ordinary Laplace transform $J^*(\theta)$ takes the form

$$J^*(\theta) = \frac{G^*(\theta)}{q+\theta},$$

with

$$G^*(\theta) \stackrel{\text{def}}{=} \int_0^\infty \exp(-\theta x + g(x))dx. \tag{B.6}$$

The estimate of $J(t)$ for large t can be derived from Tauberian theorems, remarking that the function $\exp(g(x))$ is *slowly varying* and *ultimately monotone* at infinity, according to the definitions proposed in [8]. Indeed, integrating (B.6) by parts yields

$$G^*(\theta) = \frac{1}{\theta} + \frac{M}{\sqrt{\theta + c}} + A(\theta),$$

where M is a constant and $A(\theta)$ is a function analytic in the vertical strip $-c < \Re(\theta) < 0$. The result (B.2) is obtained by a standard inversion. Details are omitted. ∎

Assume from now on condition (2.2) holds, and let $\{\widetilde{\mathcal{Q}}_n,\, n \geq 1\}$ be the sequence of *ergodic* M/M/1/∞ queues defined as follows.

- In $\widetilde{\mathcal{Q}}_n$, the intensity of the arrival process is equal to λ and the service rate has the constant value $\alpha_n = \lim_{t\to\infty} \alpha_n(t)$.
- For all n, the initial conditions of $\widetilde{\mathcal{Q}}_n$ are the same as the initial conditions of the queues $\mathcal{Q}_n$ coming in the scheme (3.20).

Letting $\widetilde{H}_n(t)$ be the probability for $\widetilde{\mathcal{Q}}_n$ to be empty at time t, we know that $\widetilde{H}_n(t)$ is driven by a Volterra integral equation, like in (3.20). Moreover, setting

$$c_n \stackrel{\text{def}}{=} \left(\sqrt{\alpha_n} - \sqrt{\lambda}\right)^2, \quad q_n \stackrel{\text{def}}{=} \mu + \frac{\lambda}{\tau \alpha_n}, \quad v_n = \min(c_n, q_n),\ \forall n \geq 0,$$

we have, for $t \to \infty$, the classical estimate, valid here since $\alpha_n > \lambda$ (see e.g. [2] p. 95),

$$\widetilde{H}_n(t) = 1 - \frac{\lambda}{\alpha_n} + h_n t^{-3/2} e^{-c_n t} + o\left(t^{-3/2} e^{-c_n t}\right), \tag{B.7}$$

where h_n is a constant, uniformly bounded in n and depending on the initial conditions. Suppose (and this is true for $n = 0$)

$$\alpha_n(t) = \alpha_n + \mathcal{O}(e^{-a_n t}), \quad \text{for some } n \geq 1.$$

Then using again monotonicity properties, together with lemma B.1 and equation (B.7), we can write

$$H_{\alpha_n}(t) = 1 - \frac{\lambda}{\alpha_n} + \mathcal{O}\big(\max(e^{-a_n t}, t^{-3/2} e^{-c_n t})\big).$$

Applying lemma B.1, and especially (B.2), to equation (3.21), we obtain immediately

$$\alpha_{n+1}(t) = \alpha_{n+1} + \mathcal{O}(e^{-a_{n+1} t}),$$

where

$$a_{n+1} = \min(a_n, v_n). \tag{B.8}$$

Exploiting the fact that v_n is a converging sequence, it follows from (B.8) that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} v_n = \min\Big(\big(\sqrt{\alpha} - \sqrt{\lambda}\big)^2, q_0\Big). \tag{B.9}$$

To prove (3.19), we introduce the iterative scheme

$$\begin{cases} \gamma_0(t) = \gamma(0) \in [0, U/\tau], \quad \forall t \geq 0\,, \\ \tau \dfrac{d\gamma_{n+1}(t)}{dt} + \gamma_{n+1}(t)\big(\mu\tau + 1 - H_{\gamma_n}(t)\big) = \mu U, \\ \gamma_{n+1}(0) \equiv \alpha(0) \in [0, U/\tau], \\ H_{\gamma_n}(t) = \psi_{\gamma_n}(t) + \int_0^t \gamma_n(s) D_{\gamma_n}(s,t) H_{\gamma_n}(s) ds, \quad n \geq 0. \end{cases} \tag{B.10}$$

This scheme differs from (3.20) or (3.23) only by the initial condition $\gamma(0)$ in the first equation of (B.10), but it turns out that this modification is substantial enough to break the monotonicity with respect to n. Nonetheless, with the notation of (3.20) and (3.23), it is straightforward to check that

$$H_{\beta_n}(t) \leq H_{\gamma_n}(t) \leq H_{\alpha_n}(t),$$

and hence $\lim_{n\to\infty} \gamma_n(t) = \alpha(t)$, uniformly. Starting from this device, we take precisely $\gamma(0) = \alpha$ in (B.10). Then (3.19) is obtained by applying (B.9) and lemma B.1. ∎

C Generators, cores and weak convergence

We quote here the material necessary for the proof of theorem 2.3. The results are borrowed from [6].

To begin with, we state the famous and fundamental Hille-Yosida theorem, which characterizes generators of strongly continuous contraction semigroups.

Theorem C.1 *A linear operator O on a Banach space $\mathcal{L}$ is closable and its closure $\overline{O}$ is the generator of a strongly continuous contraction semigroup if, and only if:*

(1) *Its domain $\mathcal{D}(O)$ is dense in $\mathcal{L}$.*

(2) *O is dissipative, i.e. $\|\lambda f - Of\| \geq \lambda \|f\|$, for every $f \in \mathcal{D}(O)$ and $\lambda > 0$.*

(3) *The range of $(\lambda - O)$ is dense in $\mathcal{L}$ for some $\lambda > 0$.*

Definition C.2 ([6], p. 17) *Let O be a closed linear operator with domain $\mathcal{D}(O)$. A subspace S of $\mathcal{D}(O)$ is said to be a* core *for O if the closure of the restriction of O to S is equal to O, i.e., if $\overline{O_{|S}} = O$.*

The next proposition ([6], p. 17), is an important criterion to characterize a core.

Theorem C.3 *Let O be the generator of a strongly continuous contraction semigroup on a Banach space $\mathcal{L}$. Then a subspace S of $\mathcal{D}(O)$ is a core if, and only if, S is dense in $\mathcal{L}$ and the range of $(\lambda - O_{|S})$ is dense in $\mathcal{L}$ for some $\lambda > 0$.*

The proof of theorem 2.3 relies heavily on the next general proposition ([6], theorem 6.1, p. 28).

Theorem C.4 *In addition to $\mathcal{L}$, let $\mathcal{L}_k, k \geq 1$, be a sequence of Banach spaces, $\Pi_k : \mathcal{L} \to \mathcal{L}_k$ be a bounded linear transformation, subject to the constraint*

$\sup_k \|\Pi_k\| < \infty$. *Let also* $\{T_k(t)\}$ *and* $\{T(t)\}$ *be strongly continuous contraction semigroups on* $\mathcal{L}_k$ *and* $\mathcal{L}$ *with generators* O_k *and* O. *We write* $f_k \to f$ *to mean exactly*

$$f \in \mathcal{L}, \quad f_k \in \mathcal{L}_k \quad \text{for } k \geq 1, \quad \text{and } \lim_{k\to\infty} \|f_k - \Pi_k f\| = 0.$$

Then, if D *is a core for* O*, the following are equivalent:*

(a) *For each* $f \in \mathcal{L}$, $T_k(t)\Pi_k f \to T(t)f$ *for all* $t \geq 0$*, uniformly on bounded intervals.*

(b) *For each* $f \in \mathcal{L}$, $T_k(t)\Pi_k f \to T(t)f$ *for all* $t \geq 0$.

(c) *For each* $f \in D$*, there exists* $f_k \in \mathcal{D}(O_k)$ *for each* $k \geq 1$*, such that* $f_k \to f$ *and* $O_k f_k \to Of$.

■

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399